MATH 207(c)–Analysis IV

1. For the following functions, determine the nature of the singularity at $z = z_0$ (i.e. regular point, pole or essential singularity), compute the residue, calculate the radius of convergence of the Laurent series.

- **(a)** $f(z) = \frac{z^2+3z+2}{z+1}$, $z_0 = -1$,
- **(b)** $f(z) = \frac{(z+1)^{\frac{1}{3}}}{z}$, $z_0 = 0$,
- **(c)** $f(z) = e^{\frac{z^2+1}{z-i}}$, $z_0 = i$,
- **(d)** $f(z) = \frac{z^{-7}+1}{1+z}$, $z_0 = 0$.

2. Let γ be a simple, closed curve in $\mathbb{C}$ which is counterclockwise oriented. What are the possible values of the following integrals, depending on the shape of γ?

- **(a)** $\int_\gamma \frac{1}{z(z+2)}\, dz$,
- **(b)** $\int_\gamma e^{\frac{1}{z^2}}\, dz$,
- **(c)** $\int_\gamma \frac{e^{iz}}{z^4+1}\, dz$,
- **(d)** $\int_\gamma \frac{\sin(z)}{z}$.

3. Let γ the circle of radius 2 centered at the origin, parametrized counter-clockwise. What is the value of the integral

$$\int_\gamma \tan(z)\, dz,$$

where, as usual, $\tan(z) = \frac{\sin(z)}{\cos(z)}$.

4. Let $\mathcal{U} \subseteq \mathbb{C}$ be an open set and $p, q : \mathcal{U} \to \mathbb{C}$ be holomorphic functions and consider the function $f(z) = \frac{p(z)}{q(z)}$ defined at the points where $q(z) \neq 0$. Let also z_0 be a point in $\mathcal{U}$ such that $q(z_0) = 0$ (i.e. a singularity of f).

- **(a)** Assume that $p(z_0) \neq 0$ and that q vanishes to first order at z_0, i.e. $q(z_0) = 0$ but $q'(z_0) \neq 0$. Show that $\mathrm{Res}_{z_0}(f) = \dfrac{p(z_0)}{q'(z_0)}$.
- **(b)** Assume that p vanishes to first order at z_0 and that q vanishes to second order at z_0, i.e. $q(z_0) = q'(z_0) = 0$ but $q''(z_0) \neq 0$. Show that $\mathrm{Res}_{z_0}(f) = \dfrac{2p'(z_0)}{q''(z_0)}$.

5. Compute the following integral:

$$\int_0^{2\pi} \frac{\cos^2(\theta)}{13 - 5\cos(2\theta)}\, d\theta.$$

Hint: Use the residue theorem, by recasting the above as a complex integral over the unit circle. For $z = e^{i\theta}$, you might need to use the identity

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\Big(z + \frac{1}{z}\Big)$$

(and similarly for $\cos(2\theta)$).